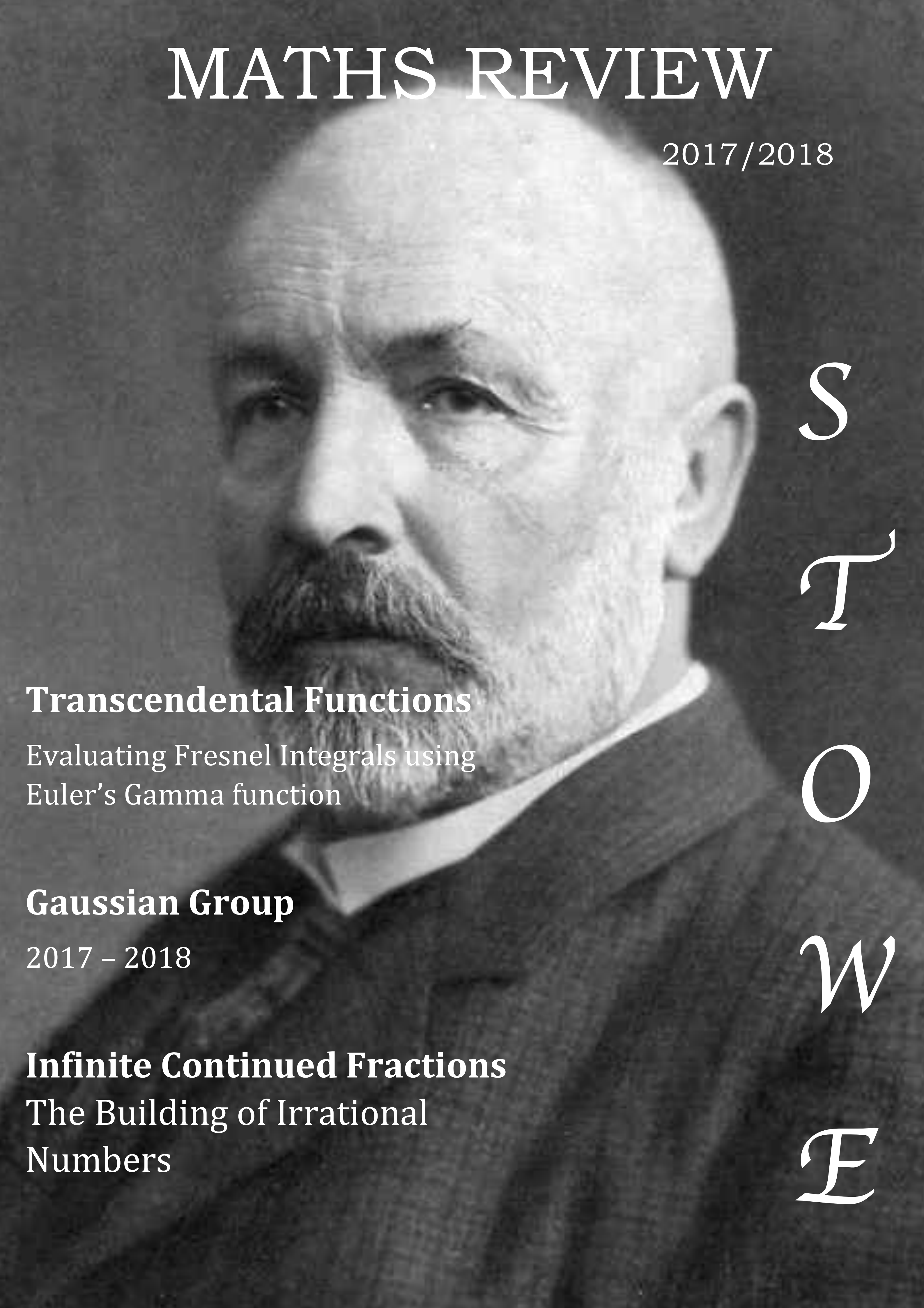


MATHS REVIEW

2017/2018

STOWE

Transcendental Functions

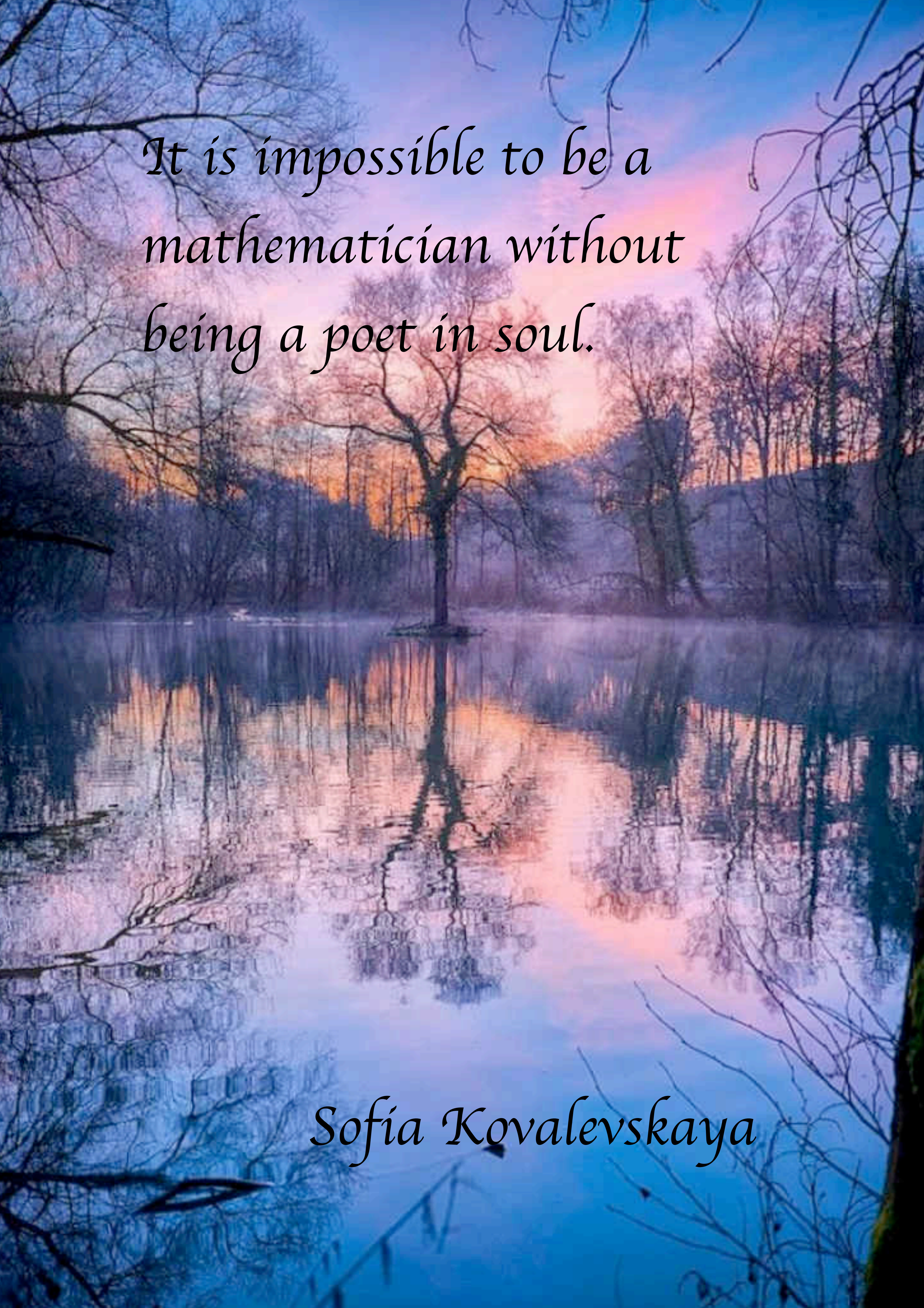
Evaluating Fresnel Integrals using Euler’s Gamma function

Gaussian Group

2017 – 2018

Infinite Continued Fractions

The Building of Irrational Numbers


It is impossible to be a
mathematician without
being a poet in soul.
Sofia Kovalevskaya

The Stowe Maths Review

The Stowe Maths Review is a magazine that gives an insight into maths at Stowe School.

Contents

Special thanks to:

All members of the Gaussian Group.

Mr Möller for his enthusiasm and full support.

All the maths teachers in the department.

Mr Karakus for his article.

Tori Roddy for helping us to publish this magazine.

Oliver Vince, the president of Gaussian Group, for his support of the maths events and society.

Mr Yadsan for designing the maths magazine.

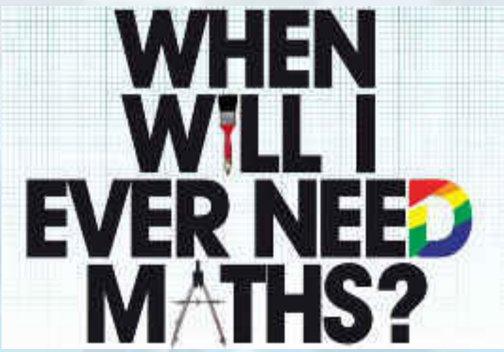
JAMES POCKLINGTON

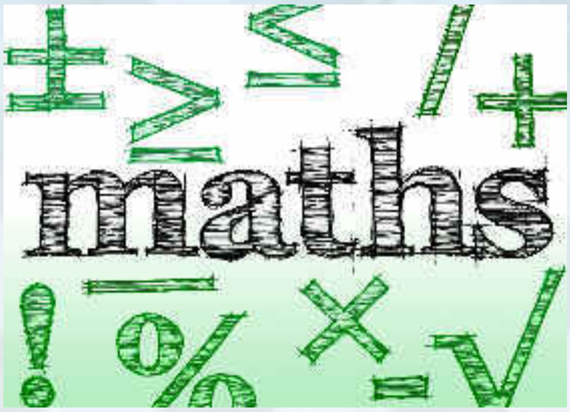
Making maths fun

(A beginner's guide to sounding smart)

When someone mentions number theory or quantum science, it all sounds quite complicated and not really accessible to someone who hasn't spent years learning about it. Well generally, that is absolutely correct! HOWEVER, given just a little time and a few examples the following pages should explain enough maths so that you too can pretend to know what a complex number is, and why the meaning of the universe really is 42, (okay maybe not that but at least we can find the square root of -1).

Right then, to get started let's have a look at how maths can be broken fairly easily, a pretty simple way of doing this is by proving that 2=1, sounds pretty dumb right? Well, that's because it is; however it makes for a great party trick if you're okay with looking like a bit of a geek.





Prove: $1=2$

Proof:

1, let $A=B$

2, multiply by (A), therefore $A^2=AB$

3, subtract B^2, therefore $A^2-B^2=AB-B^2$

4, Factorise, $(A+B)(A-B)=B(A-B)$

5, Divide by (A-B), $(A+B)=B$

6, since $A=B$ this is the same as…

$B+B=B$… or $2B=B$

7, Divide by B and we get $2=1$

Alright, obviously that's not strictly true since… well it isn't, the real reason is that you can't divide by A-B because that would equal 0 and mathematically you cannot divide by 0.

Right then, now that we're familiar with how maths can be broken and doesn't make a huge amount of sense if you start to make a few 'small' assumptions. Well now let's try to take that one step further. When posed with the following question, what would you assume the most logical answer to be?

"What would the sum of all positive integers, i.e. from 1 to infinity equal?"

Naturally, it would be easy just to say, "Infinity plus a bit more" or "A really big number". Well, unfortunately, there is an actual solution at the bottom of the page, but feel free to try to work it out before seeing the answer first.

Okay, so we are going to need to generate some sequences since infinity can't really be accessed that easily.

This side of maths is mean and unpredictable at best. Now to back up my baseless statement, consider the sequence 1+1+1+1+… going on indefinitely it would equal infinity right? Right. Now consider 1-1+1-1+… (S1) technically to go infinitely here you would get an answer of 1 or 0 so to keep things simple let's just say the sequence equals the average of ½ Next let's imagine 1-2+3-4+… (S2) but now double it to 2(S2) but think of it like this.

1-2+3-4…

1-2+3-4+5… +

=1-1+1-1+…. Which again equals ½. Therefore (S2) = ¼.

Lastly, let's use 1+2+3+4… (S) and attempt (S)-(S2)

=1+2+3+4+5+6……

- (1-2+3-4+5-6…

= (1-1) + (2--2) + (3-3)…

=0+4+0+8+0+12+…

Therefore if we take a factor of 4 from all of these values, we get:

(S)-(S2) =4(1+2+3+4+5…)

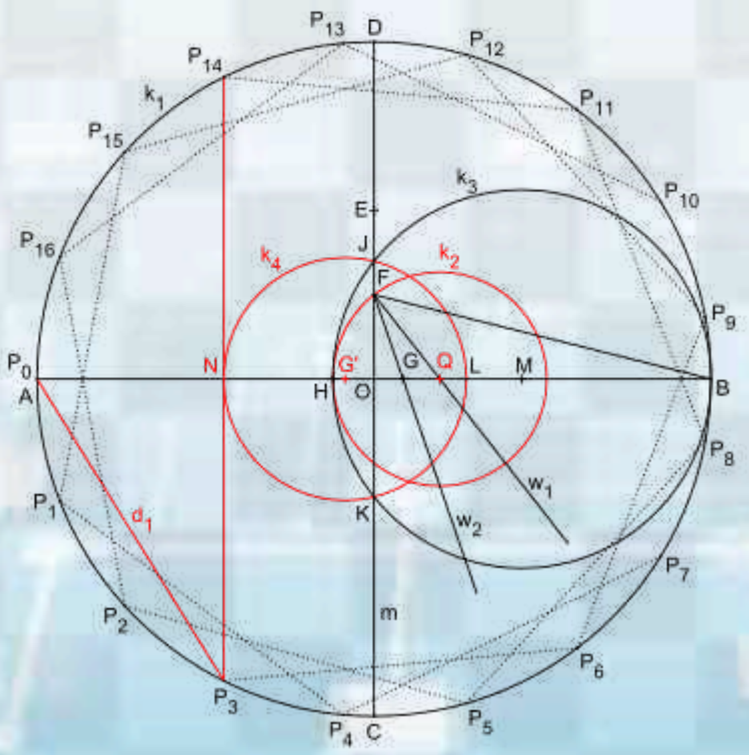
Which means that: (S)-¼=4(S)

A little algebraic manipulation and we're finished with 3(S) = -1/4

so by simply

dividing both sides by three we can see that (S) = -1/12.

Right then, everyone right from the very beginning of education, knows about shapes… 2D ones specifically, many have specific constructions to be drawn accurately, such as the triangle with three circles or arcs that all meet at one point, with the centres being the vertices of the final shape. However, this is not possible for all known 2D shapes, in fact, the largest known version of this is… the "Heptadecagon" I.E the seventeen sided regular shape.



There's a lot there to pay attention to so I am going to try to keep the explanation brief, in terms of real world application this isn't really useful anywhere, mostly just a cool shape that was researched 'just because we can' Anyway, the whole point is that we are trying to find set points on the shape such as the two vertical lines or the bottom left diagonal, as they make up exactly 'x' seventeenths of the shapes circumference, to find a proper step by step guide on how this is achieved then check out the video by 'Numberphile' as they can show this off far more eloquently than I can.

Although there isn't much use for this side of maths, does that really matter? Most of why we do things is just because we can, and well if that isn't a good enough reason and at least a little bit interesting, then I don't know what could possibly be.

Okay then, for those of you who are avid fans of the Hitchhikers Guide to the Galaxy, you may recognise one of Marvin's' more notable accomplishments, discovering the square root of -1, but unfortunately, in the modern day, such an achievement is taught in A Level Further Maths. The exact topic is mostly used to explain complex numbers, but why do we care about them? Well, mostly we don't… In the 16th century Italy, if a mathematician wanted to apply for the job of professor of mathematics (or similar) instead of handing in a CV and going to an interview they would simply have a 'duel' with the current professor. The duels were not fought with guns; however, instead it was with pure grey-matter, i.e. I can solve these ten questions and bet that I can solve your ten better and faster than you can mine. Naturally, as people got smarter over the years, it was important to these people to be able to solve increasingly difficult questions so that they had a 'Stronger weapon in their arsenal'.

So where does the square root of -1 come in?

Consider the simple equation $X^2 + 1 = 0$

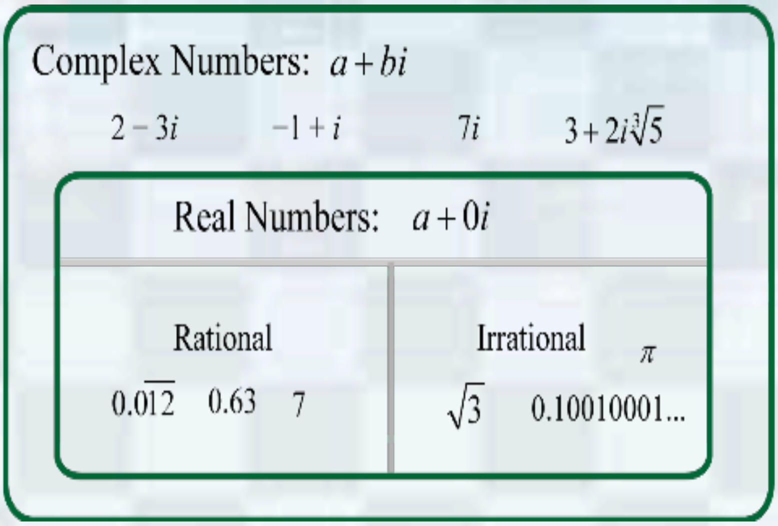
Quickly we see that $X^2 = -1$

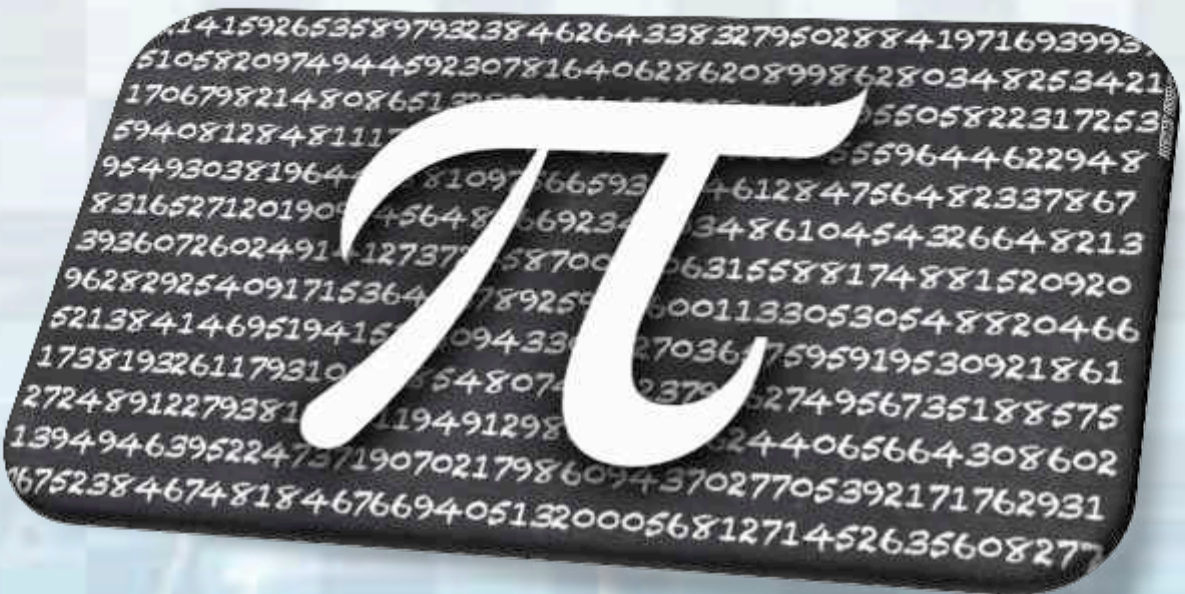
And therefore X = root -1

Which according to all common logic is incorrect, since X^2 would then equal =?

This means mathematicians needed something else that made them seem just a bit smarter, and thus the introduction of ‘i’

Not because we wanted to explore this side of maths, but because we accidentally ended up here instead.





REMEMBERING MARYAM MIRZAKHAN

Maryam was born and raised in Iran. During her teenage years she took part in International Mathematical Olympiads, becoming the first Iranian student to achieve a perfect score and win two gold medals. In 2014 Maryam Mirzakhani was the *first ever female mathematician* to receive the Fields Medal, which is sometimes described as a Nobel prize for mathematics.

Maryam said the most rewarding moments in her life had been the "Aha" moment, the excitement of discovery and enjoyment of understanding something new – the feeling of being on top of a hill and having a clear view. But most of the time, doing mathematics for me is like being on a long hike with no trail and no end in sight.

<u>The beauty of mathematics only shows itself to more patient followers.</u>

She became a mathematics Professor at Stanford University in the U.S. and is considered to be one of the greatest mathematicians of recent times.

Maryam described how her love of mathematics grew from her teenage years onwards: ‘The more time I spent on maths, the more excited I got’ Sadly Maryam Mirzakhani died of breast cancer in 2017 aged just 40 years old.

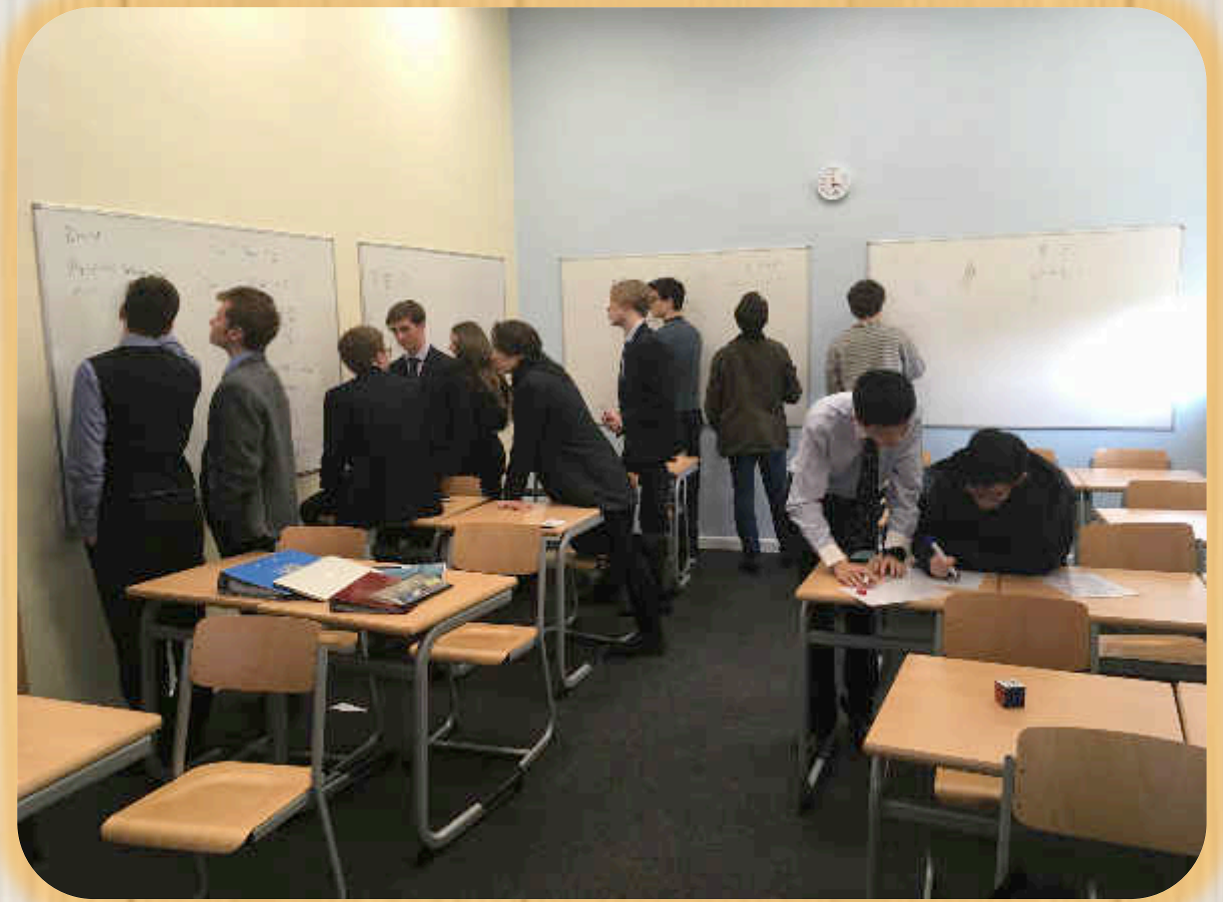
Another exciting year for the mathematicians of Stowe:

Vertical Stretch Sessions

Upper School

We have not succeeded in answering all our problems. The answers we have found only serve to raise a whole set of new questions. In some ways, we feel we are as confused as ever, but we believe we are confused on a higher level and about more important questions.

Vertical Stretch Sessions (VSS) were one of many academic activities offered in the maths department this year. This activity was an opportunity for students to understand mathematics at a deeper level by making 'how serve why'. That is, as well as learning how to use mathematics; we also strived to understand why things work in maths.



From September to January, in VSS, we studied a course on Number Theory. This course included: Using Euclid's algorithm to solve integer equations. Prime and co-prime numbers, Euler's totient function. The fundamental theorem of arithmetic, unique factorisation domain (UFD) and we scratched the surface of non-UFD. Divisibility, congruence relations, residue classes and modular arithmetic. We then moved onto linear Diophantine equations and general solutions to integer equations.

The second course in VSS was 'Mathematics of Quantum Mechanics'. In this course, we studied the statistical interpretation of QM (the modulus square of the wave function is the probability). This course included: normalizing the wave function and finding the normalisation constant. We further went into the postulates of QM and the momentum operator in quantum mechanics in position basis. Sketching the wave function, evaluating the expectation values for the position and the momentum. We then moved onto uncertainties in x and p, and how to apply Heisenberg's uncertainty principle (position and momentum) in a given problem and understand the nature of canonical conjugation.

To be able to tackle and solve Schrodinger's equation, we studied partial differentiation and partial differential equations. Focusing on the separation of variables, we solved Time-Independent Schrodinger's equation (TISE) for a particle in an infinite potential well and obtained energy Eigen-states.

Lastly, after completing the course on QM, we studied Indeterminate Equations. In this course, we learnt to solve equations using 'clever tricks'. These included factorisation, inequalities, quadratic and Euclidian methods.

$$a_f(n) \sim \frac{(-1)^{n-1}}{2\sqrt{n-\frac{1}{24}}} \cdot e^{\pi\sqrt{\frac{n}{6}-\frac{1}{144}}}.$$

VSS Lower School:

Lower School mathematicians have covered various topics in pure mathematics this year. In Series and Sequences course, we covered convergent and divergent sequences, strictly increasing, and non-decreasing, decreasing and monotonic sequences. In series, we investigated geometric sequences and condition for convergence. We further explored how to use integral test and p-series to identify bounded-ness.

Lower school mathematicians then studied alternating, harmonic, convergent, divergent sequences and series. Moreover, we analysed the comparison test to see how one can use a known series to work out an unknown series. We finished this course by exploring the Ratio tests.

Pythagorean Discovery of Infinite Continued Fractions of Irrational Numbers

The **Babylonian** clay tablet (c.1800-1600 BC) gives an approximation of $\sqrt{2}$ in four sexagesimal figures, 1, 24, 51, 10, which is accurate to about six decimal digits and it is the closest possible three-place sexagesimal representation of $\sqrt{2}$:

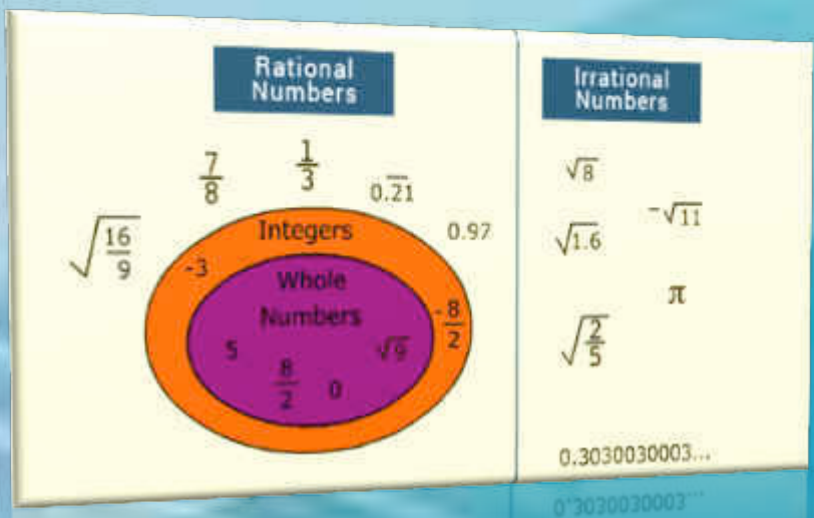
$$1+\frac{24}{60}+\frac{51}{60^2}+\frac{10}{60^3}=\frac{30547}{21600}=1.41421\overline{296}.$$

Another early close approximation is given in **ancient Indian** mathematical texts, the **Sulbasutras** (c. 800-200BC) as follows: *Increase the length (of the side) by its third and this third by its own fourth less the thirty-fourth part of that fourth*. That is

$$1+\frac{1}{3}+\frac{1}{3\times4}-\frac{1}{3\times4\times34}=\frac{577}{408}=1.41421\overline{56862745098039}$$

INTRODUCTION

Examples of Rational and Irrational Numbers



Definition: Let the **limit** of the sequence to be the value of the infinite continued fraction $[\boldsymbol{a_0};\ \boldsymbol{a_1}, \boldsymbol{a_2}, \dots]$

Note that $[a_0;\ a_1, a_2, \ldots =]\ a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{}}}$

This definition gives meaning to the notion of an "infinite continued fraction". We only ever consider infinite continued fractions where entries satisfy

- a_i is an integer for all i;
- $a_i > 0$ for all $i \geq 1.$

For example, if $x_n = [2; 2, 2, 2, \ldots, 2]$ (with $n + 1, 2s$), the we have

$$x_n = 2 + \frac{1}{x_{n-1}}.$$

We know that $\lim_{n\to\infty} x_n$ exists; let it be u. Then clearly $\lim_{n\to\infty} x_{n-1} = u,$ so by properties of limits we have

$$u = 2 + \frac{1}{u}.$$

Examples:

1) What is the continued fraction expansion of π ?
We apply the continued fraction algorithm. A few minutes' to work with a calculator shows that, to a few places of decimals,

$$\pi = 3.141592653589793,$$

$$\frac{1}{0.141592653589793} = 7.062513305931046$$

$$\frac{1}{0.062513305931046} = 15.99659440668572$$

$$\frac{1}{0.99659440668572} = 1.003417231013372$$

$$\frac{1}{0.003417231013372} = 292.6345910144503$$

$$\frac{1}{0.6345910144503} = 1.575818089492172$$

.
.
.

… so the continued fraction for π begins

$$\pi = [3; 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, \dots] = 3 + \cfrac{1}{7+\cfrac{1}{15+\cfrac{1}{1+\cfrac{1}{292+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\cdots}}}}}}}}}$$

Another representation of $\pi = \cfrac{4}{1+\cfrac{1^2}{3+\cfrac{2^2}{5+\cfrac{3^2}{7+\cfrac{4^3}{9+\cdots}}}}}$

2) $\sqrt{2} = [1; 2, 2, 2, \dots] = [1; \overline{2}] = 1 + \cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cdots}}}}$

$\overline{2}$ means repeating

How do we show without calculator that $\sqrt{2} = 1 + \cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cdots}}}}$

PROOF:

$$\sqrt{2} = 1 + (\sqrt{2} - 1) \quad \textbf{eqn1}$$

$$\sqrt{2} - 1 = \frac{\sqrt{2} - 1}{1}$$

Rationalise the numerator

$$\sqrt{2} - 1 = \frac{\sqrt{2}-1}{1} \times \frac{\sqrt{2}+1}{\sqrt{2}+1} = \frac{1}{1+\sqrt{2}} \quad \textbf{eqn 2}$$

Substitute eqn 2 into eqn 1

$$\sqrt{2} = 1 + \left(\sqrt{2} - 1\right) = 1 + \frac{1}{1+\sqrt{2}} \quad \textbf{eqn 3}$$

Hence

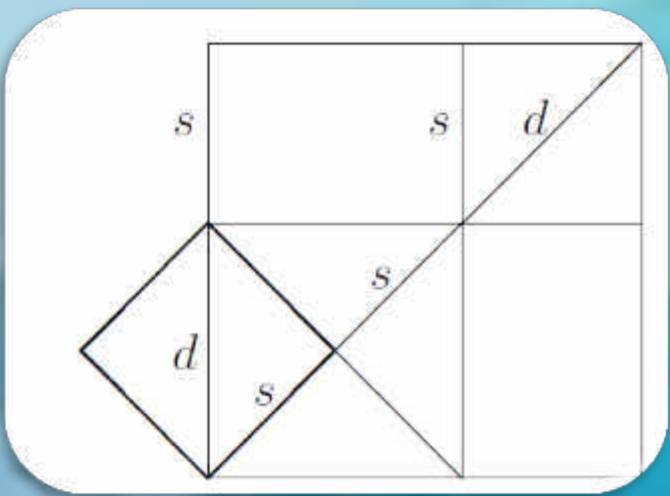
$$\sqrt{2} = 1 + \frac{1}{1+\sqrt{2}} = 1.41421356\dots$$

Replacing for $\sqrt{2}$ repeatedly in equation 3 we obtain the following

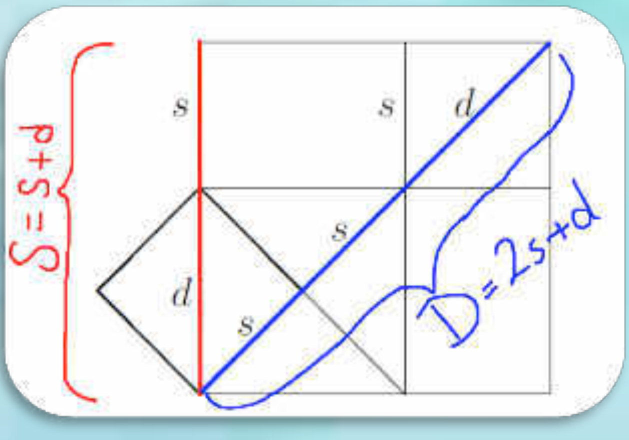
$$\sqrt{2} = 1 + \frac{1}{1+\sqrt{2}} = 1 + \cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cdots}}}}$$

How the Pythagoreans knew that the ratio of the diagonal to the side of a square is irrational?

The Pythagoreans knew that the ratio of the diagonal to the side of a square is irrational. According to the historian of mathematics David Fowler, they may have reasoned something like this.



Let **s** and **d** be the side and **diagonal lengths** of a **square**. **Rotate** the square through **45 degrees**. Prolong the diagonal by $\boldsymbol{s}$ and draw a new square on this side, with side and diagonal lengths $\boldsymbol{S}$ and $\boldsymbol{D}$**,** respectively. We see from the figure below that $\boldsymbol{S = s + d}$, and $\boldsymbol{D = 2s + d}$; so



$$\frac{S+D}{S} = \frac{3s+2d}{s+d} = 2 + \frac{s}{s+d}$$

Let $u = \frac{s+d}{s}$. Hence, $\frac{1}{u} = \frac{s}{s+d}$. Since any two squares are similar, we also have $u = \frac{S+D}{S}$, and so

$$u = 2 + \frac{1}{u}.$$

Substituting this expression for u into the right-hand side of the expression repeatedly, we see that

$$u = 2 + \cfrac{1}{2 + \cfrac{1}{u}} = 2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{u}}} = \cdots$$

If u were a rational number, then the procedure for finding the continued fraction for u terminates, and the continued fraction is finite. The above argument shows that, if this algorithm is applied to *irrational* u, it never terminates; the algorithm “spins its wheels" and the next number at each stage is always u. You might guess that this means u can be expressed as an ‘infinite continued fraction’.

$$u = \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cdots}}}.$$

Hence, every real irrational number has a unique expression as an infinite continued fraction.

PRACTICE FOR THE READERS:

Find continued fraction of the following

1) $\sqrt{11}$
2) Golden Ratio : $\varphi = \frac{1+\sqrt{5}}{2}$
3) e

To accelerate and challenge the brightest mathematicians at Stowe the maths department offered two maths projects this year. From the foundations of mathematics to number theory; this year's projects varied from Cantor's Infinite set theory (leading to Gödel's theorem) to Riemann Zeta function.

Stoics, who intend to study mathematics and mathematical sciences at the world's leading universities, were invited to the Local Maximum Sessions (LMS). In LMS stoics tackled Oxbridge entry papers and World Maths Olympiad questions. In these sessions, as well as solving challenging questions, we studied inequalities that are used in the Olympiads. This includes Arithmetic Mean, Geometric Mean (AM-GM) inequality, the Cauchy-Schwarz Inequality and some other useful inequalities obtained from convex functions. In these sessions, we focused on 'questioning the question to find the best starting point to answer the question'.

WEI LANG ZHAO

CAUCHY-BUNIAKOWSKY-SCHWARZ INEQUALITY

$$|(\alpha, \beta)| \leq |\alpha||B|$$

Let $f(x) = (a_1x - b_1)^2 + (a_2x - b_2)^2 + \cdots + (a_nx - b_n)^2$

We can deduce that

$$f(x) \geq 0.$$

Then the discriminant of f(x) is either zero or less than zero:

$$\Delta = 4(a_1b, +a_2b_2 + \cdots a_nb_n)^2 - 4(a_1^2 + a_2^2 + \cdots \cdot a_n^2)^2(b_1^2 + b_2^2 + \cdots b_n^2) \leq 0$$

$$(a_1b_1 + a_2b_2 + \cdots a_nb_n)^2 \leq \left(a_1^2 + a_2^2 + \cdots a_n^2\right)(b_1^2 + b_2^2 + \cdots b_n^2)$$

When $a_ix - b_i = 0$ then the equality holds

So $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \cdots = \frac{a_n}{b_n}$

Integral Form

Let $\varphi(t) = t^2 \int_a^b f^2(x)dx + 2t \int_a^b f(x)g(x)\,dx + \int_a^b g^2(x)\,dx$

$$\int_a^b [t^2f^2(x) + 2tf(x)g(x) + g^2(x)]\,dx = \int_a^b [tf(x) + g(x)]^2\,dx \geq 0$$

So this quadratic function only have 1 point on x-axis at most $\Delta \leq 0$

$$\left(2\int_a^b f(x)g(x)\,dx\right)^2 - 4\int_a^b f^2(x)\,dx\int_a^b g^2(x)\,dx \le 0$$

$$\left(\int f(x)y(x)\,dx\right)^2 \le \int f^2(x)\,dx\int g^2(x)\,dxf$$

Only when f(x) and g(x) are linear relationship then the equality holds

Useful definitions and applications of C-B-S inequality

Vectors

Let a and b be two vectors in R^n

The Cauchy-Schwartz inequality states that |a · b| ≤ |a||b|

2 - Dimensions

$$(a^2 + b^2)(c^2 + d^2)$$

$$= a^2 \times c^2 + b^2 \times d^2 + a^2 \times d^2 + b^2 \times c^2$$

$$= (ac + bd)^2 + (ad - bc)^2 \ge (ac + bd)^2$$

$$a, b, c, d \in R$$

General form in 2D $\sum_{i=0}^{n} {a_i}^2 \sum_{i=1}^{n} {b_i}^2 \ge [\sum_{i=1}^{n} a_i b_i]^2$

Triangular form

$$\left(\sqrt{a^2+b^2}+\sqrt{c^2+d^2}\right)^2$$

$$a^2+b^2+c^2+d^2+2\sqrt{a^{-2}+b^2}\times\sqrt{c^2+d^2}$$

$$\geq a^2+b^2+C^2+d^2+2|ac+bd|$$

$$=a^2+2ac+c^2+b^2+2bd+d^2$$

$$\sqrt{a^2+b^2}+\sqrt{c^2+d^2}\geq\sqrt{(a+c)^2+(b+d)^2}$$

How to use Cauchy-Buniakowsky-Schwarz Inequality

If a,b,c is a positive number

$$\frac{2}{a+b}+\frac{2}{b+c}+\frac{2}{c+a}>\frac{9}{a+b+c}$$

Prove

$$(a+b+b+c+c+a)\times\left[\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}\right]$$

$$\left[\left(\sqrt{a+b}\right)^2+\left(\sqrt{b+c}\right)^2+\left(\sqrt{a+c}\right)^2\right]\left[\left(\sqrt{\frac{1}{a+b}}\right)^2+\left(\sqrt{\frac{1}{b+c}}\right)^2+\left(\sqrt{\frac{1}{a+c}}\right)^2\right]$$

$$\geq\left(\sqrt{a+b}\sqrt{\frac{1}{a+b}}+\sqrt{b+c}\sqrt{\frac{1}{b+c}}+\sqrt{a+c}\sqrt{\frac{1}{a+c}}\right)^2$$

$$=(1+1+1)^2=9$$

GAUSSIAN GROUP 2017-2018

Stowe's maths society, Gaussian Group, had several meetings this year.

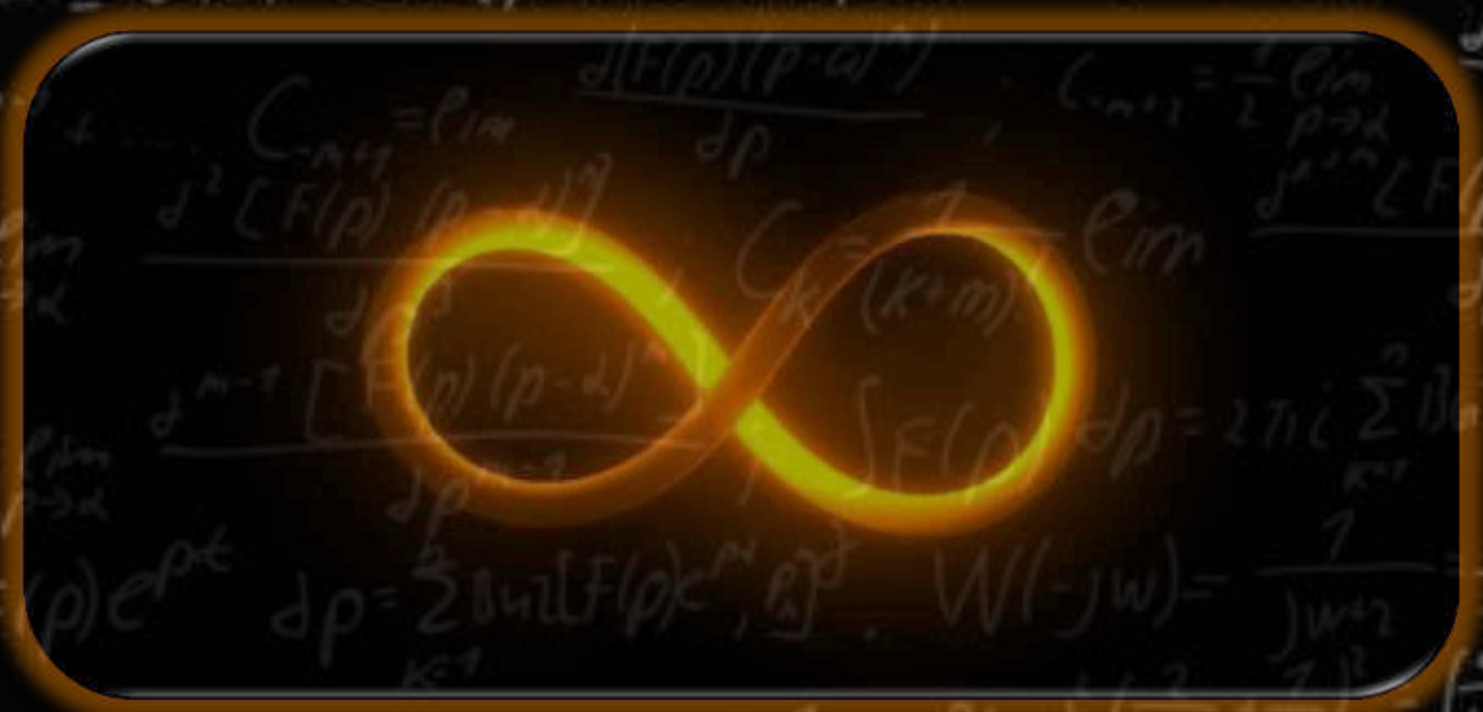
‘GOD AT THE SUBATOMIC LEVEL’

Can matter create matter? Can energy create energy? Can the equation of gravity create gravity? Can something come from nothing? Is it creation or is it simply a product of mindless, unguided, blind process, which happened by chance? Am I a lottery ticket? Is it God or Randomness? Even if we know how we are, even if we know why we are, the fact that we are is a mystery. Which direction does science point at in this mystery? Which one does science favour? God or Randomness. Quantum Mechanics is the strongest theory that has ever been constructed. Interpretation of the unknown is a difficult task. However, Quantum Mechanics seems to be the best candidate to be our interpreter. It claims to see unseen and knows unknown. What does Quantum Mechanics tell us about the creator? The first Gaussian Group meeting was titled ‘God at the subatomic level’. In this talk, Mr Yadsan showed the strong connection between mathematics, physics, theology and philosophy. The talk was inspiring and thought-provoking. During this talk, profound questions were asked, and subtle discussions took place.

Student Presentations

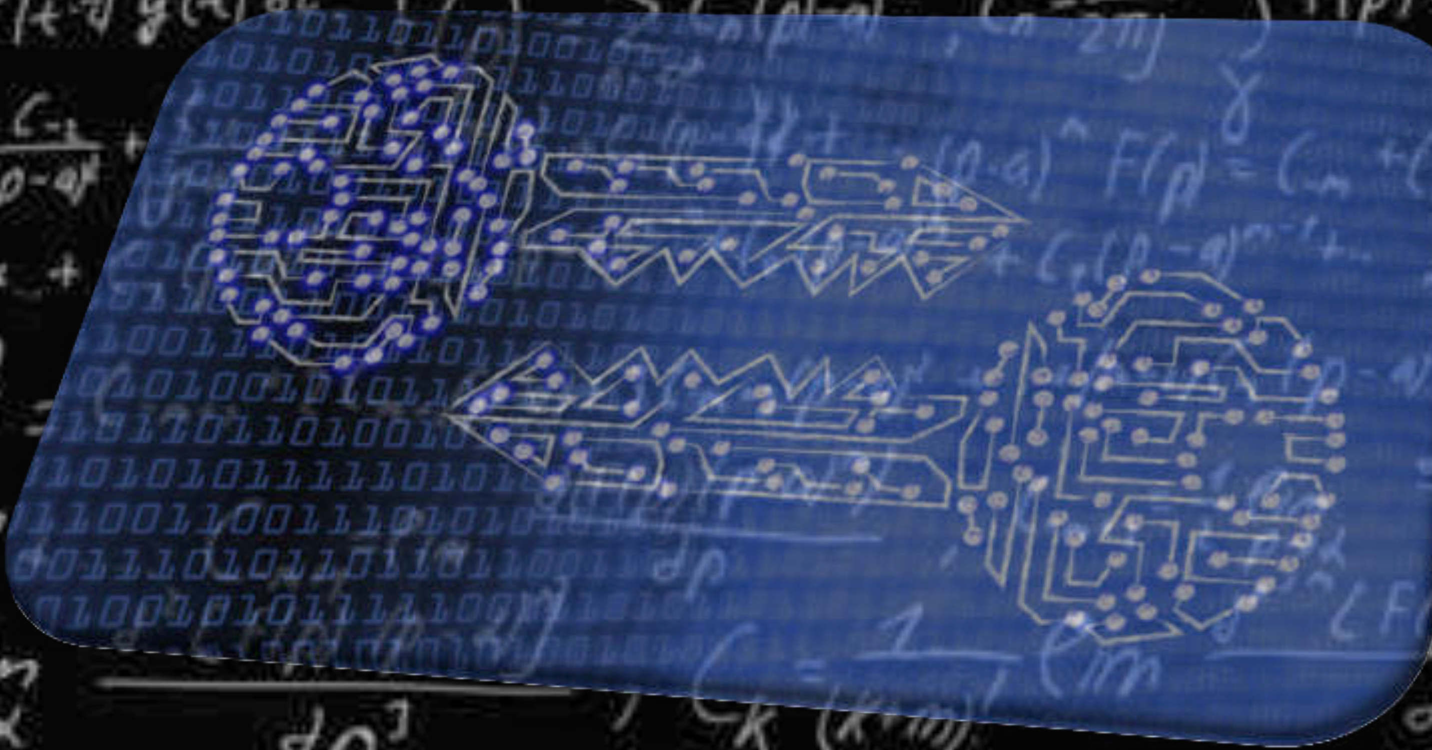
Gaussian Group hosted many exciting student presentations this year. Hugo Robinson delivered the first talk 'On the development of weapons; from bows and arrows to muskets using maths'. This was a fascinating talk explaining historical aspects of how weapons evolved and what mathematics has got to do with it.

The second talk was "Cryptography, the art of writing or solving codes.' by Oliver Vince. The talk was inspiring and subtle. Oliver highlighted the use of modular arithmetic in cryptography; the process of coding and decoding and how to secure information in this digital age.



Another exciting talk was 'Infinity' by James Pocklington. Infinity has been puzzling the best mathematicians for centuries. In his talk, James thoroughly questioned the nature of infinity and gave mind-blowing examples, such as Hilbert's Hotel and irrational numbers. This was a profoundly thought-provoking talk.

The next talk was delivered by Freddie Woods on Algorithms and cool examples! This was an interesting talk on cryptography and the nature of information. The talk showed the uses of algorithms, how to rate an algorithm and some cool examples. This talk showed the deep connection between mathematics and computing.



The final talk of the year was "Banach-Tarski Paradox" by William Fox. William shone a light on one of the most puzzling paradoxes in the history of mathematics. The nature of uncountable infinities is fascinating and mind-boggling at the same time.

One of the highlights of this year has been inviting the local schools to a Gaussian Group meeting. To promote love of mathematics and to inspire young people, the Gaussian Group strives to establish a network of mathematicians with the local schools. We look forward to meeting with local schools in the near future again.

We thank all our members and supporters.

We wish a great success to Oliver Vince (President of Gaussian Group) and James Pocklington, who have chosen to study mathematics at university.

Fresnel Integrals

Consider the Integrals

$$\int_0^{\infty} \cos(s^2)\, ds = A_1 \quad \int_0^{\infty} \sin(s^2)\, ds = A_2$$

These are two extremely important integrals in physics. One way to calculate such integrals is to consider Gamma function, one of Euler's remarkable findings:

$\Gamma(n) = \int_0^{\infty} e^{-x}x^{n-1}dx$, If you then use the integration by parts formula $\int udv = uv - \int vdu$ for the case $\Gamma(n+1)$, Letting u=x^n and dv=$e^{-x}dx$

$$\Gamma(n+1) = \int_0^{\infty} e^{-x}x^n dx = -e^{-x}x^n|_0^{\infty}$$
$$-\int_0^{\infty} -e^{-x}nx^{n-1}\, dx = n\int_0^{\infty} -e^{-x}x^{n-1}dx$$

Therefore

as $\Gamma(n) = \int_0^{\infty} e^{-x}x^{n-1}dx$, then $\Gamma(n+1) = n\Gamma(n)$

As $\Gamma(1)$=$\int_0^{\infty} e^{-x}x^{1-1}dx$=$\int_0^{\infty} e^{-x}dx$=$-e^{-x}|_0^{\infty}$ = 1

$$\Gamma(1) = 1$$
$$\Gamma(1+1) = \Gamma(2) = 1\Gamma(1) = 1 = 1!$$
$$\Gamma(2+1) = \Gamma(3) = 2\Gamma(2) = 1!\times 2 = 2!$$
$$\Gamma(3+1) = \Gamma(4) = 3\Gamma(3) = 2!\times 3 = 3!$$

....

Therefore it can be concluded that $\Gamma(n) = (n-1)!$. This equality also allows for the possibility of fractional and negative factorials:

$$\Gamma\left(\frac{1}{2}\right) = \left(-\frac{1}{2}\right)!$$

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty e^{-x}x^{-\frac{1}{2}}dx = \int_0^\infty \frac{e^{-x}}{\sqrt{x}}dx$$

Let $x = t^2, dx = 2tdt$:

$$\int_0^\infty \frac{e^{-t^2}}{\sqrt{t^2}}2dt = 2\int_0^\infty e^{-t^2}dt = 2I$$

Where $I = \int_0^\infty e^{-u^2}du = \int_0^\infty e^{-v^2}dv$

$$I^2 = \left(\int_0^\infty e^{-u^2}du\right)\left(\int_0^\infty e^{-v^2}dv\right)$$

$$= \int_0^\infty \int_0^\infty e^{-u^2}e^{-v^2}dudv = \int_0^\infty \int_0^\infty e^{-(u^2+v^2)}dudv$$

As $dudv$ is simply an area in Cartesian form over the first quadrant. Changing Cartesian to polar form using the substitutions, $u = rcos(\theta), v = rsin(\theta)$, the area becomes $rdrd\theta$. r's scale can run from ∞ to 0 and θ's scale runs from 0 to $\frac{\pi}{2}$ to cover the whole first quadrant. As $u^2 + v^2 = r^2$ you can form the integral.

$$\int_0^{\frac{\pi}{2}}\int_0^\infty e^{-r^2}rdrd\theta$$

do the interior integral first

$$\int_0^\infty e^{-r^2}rdr = -\frac{1}{2}e^{-r^2}|_0^\infty = 0 - -\frac{1}{2} = \frac{1}{2}$$

Substituting this back into are original integral you get:

$$\int_0^{\frac{\pi}{2}} \frac{1}{2} d\theta = \frac{1}{2}\theta|_0^{\frac{\pi}{2}} = \frac{\pi}{4} - 0 = \frac{\pi}{4} = I^2$$

Therefore

$$I = \frac{1}{2}\sqrt{\pi}$$

$$\Gamma\left(\frac{1}{2}\right) = 2I = \sqrt{\pi} = \left(-\frac{1}{2}\right)!$$

To extend the Gamma Function to complex numbers you can use the substation for the dummy variable $x = y(p + iq), dx = (p + iq)dy$ Where p and q are both real numbers:

$$\Gamma(n) = \int_0^{\infty} e^{-y(p+iq)}\left(y(p + iq)\right)^{n-1}(p + iq)dy$$

$$= \int_0^{\infty} (p + iq)^n y^{n-1} e^{-py} e^{-iqy} dy$$

$$\Gamma(n) = (p + iq)^n \int_0^{\infty} y^{n-1} e^{-py} e^{-iqy} dy$$

therefore

$$\frac{\Gamma(n)}{(p + iq)^n} = \int_0^{\infty} y^{n-1} e^{-py} e^{-iqy} dy$$

Then if you write $p + iq$ in polar form, Where $r = \sqrt{p^2 + q^2}$, $\alpha = \tan^{-1}\left(\frac{q}{p}\right)$

$$p + iq = r(cos(\alpha) + isin(\alpha) = re^{i\alpha}$$

Therefore

$$\int_0^{\infty} y^{n-1} e^{-py} e^{-iqy} dy = \frac{\Gamma(n)}{(re^{i\alpha})^n} = \frac{\Gamma(n)}{e^{i\alpha n} r^n} = \frac{\Gamma(n)}{r^n} e^{-i\alpha n}$$

Expanding the last and first terms with Euler's identity,

$$\int_0^\infty y^{n-1}e^{-py}e^{-iqy}dy = \int_0^\infty y^{n-1}e^{-py}(cos(-qy) + i\sin(-qy))dy$$
$$= \frac{\Gamma(n)}{r^n}e^{-i\alpha n} = \frac{\Gamma(n)}{r^n}(cos(-\alpha n) + isin(-\alpha n))$$

As the real and imaginary parts must be equal you can split the equation above into two separate equations:
The Real parts:

$$\int_0^\infty y^{n-1}e^{-py}\,cos(-qy)dy \;= \frac{\Gamma(n)}{r^n}cos(-\alpha n)$$
$$\int_0^\infty y^{n-1}e^{-py}\,cos(qy)\,\mathrm{dy} \;= \frac{\Gamma(n)}{r^n}cos(\alpha n)$$

The complex parts:

$$\int_0^\infty y^{n-1}e^{-py}(cos(-qy) + i\sin(-qy)dy$$
$$= \frac{\Gamma(n)}{r^n}(cos(-\alpha n) + isin(-\alpha n))$$
$$\int_0^\infty y^{n-1}e^{-py}\sin(-qy)dy = \frac{\Gamma(n)}{r^n}sin(-\alpha n))$$
$$\int_0^\infty -y^{n-1}e^{-py}\sin(qy)dy = -\frac{\Gamma(n)}{r^n}sin(\alpha n))$$
$$\int_0^\infty y^{n-1}e^{-py}\sin(qy)dy = \frac{\Gamma(n)}{r^n}sin(\alpha n))$$

Now if you let $n = \frac{1}{2}, p = 0, q = 1$
as $\alpha = \tan^{-1}\left(\frac{q}{p}\right)$,then

$$\alpha = \tan^{-1}\left(\frac{1}{0}\right) = \tan^{-1}(\infty) = \frac{\pi}{2}$$

and as $r = \sqrt{p^2 + q^2}$,then

$$r = \sqrt{0^2 + 1^2} = 1.$$

Then substituting these numbers into the final equations above gives:

$$\int_0^\infty y^{-\frac{1}{2}} cos(y)\,\mathrm{dy} \ = \frac{\Gamma\left(\frac{1}{2}\right)}{1} cos\left(\frac{\pi}{2} \times \frac{1}{2}\right)$$

$$\int_0^\infty y^{-\frac{1}{2}} \sin\,(y) dy = \frac{\Gamma\left(\frac{1}{2}\right)}{1} sin\left(\frac{\pi}{2} \times \frac{1}{2}\right)$$

As I calculated earlier $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

$$\Gamma\left(\frac{1}{2}\right) sin\left(\frac{\pi}{4}\right) = \sqrt{\frac{\pi}{2}}$$

$$\Gamma\left(\frac{1}{2}\right) cos\left(\frac{\pi}{4}\right) = \sqrt{\frac{\pi}{2}}$$

As

$$\int_0^\infty y^{-\frac{1}{2}} cos(y)\,\mathrm{dy} \ = \int_0^\infty \frac{cos(y)}{\sqrt{y}}\mathrm{dy} \ = \sqrt{\frac{\pi}{2}}$$

$$\int_0^\infty y^{-\frac{1}{2}} sin(y)\,\mathrm{dy} \ = \int_0^\infty \frac{sin(y)}{\sqrt{y}}\mathrm{dy} \ = \sqrt{\frac{\pi}{2}}$$

If you then substitute $y = s^2, dy = 2s,$

$$\int_0^\infty \frac{cos(s^2)}{s} 2sds \ = \int_0^\infty \cos\,(s^2)2ds \ = \sqrt{\frac{\pi}{2}}$$

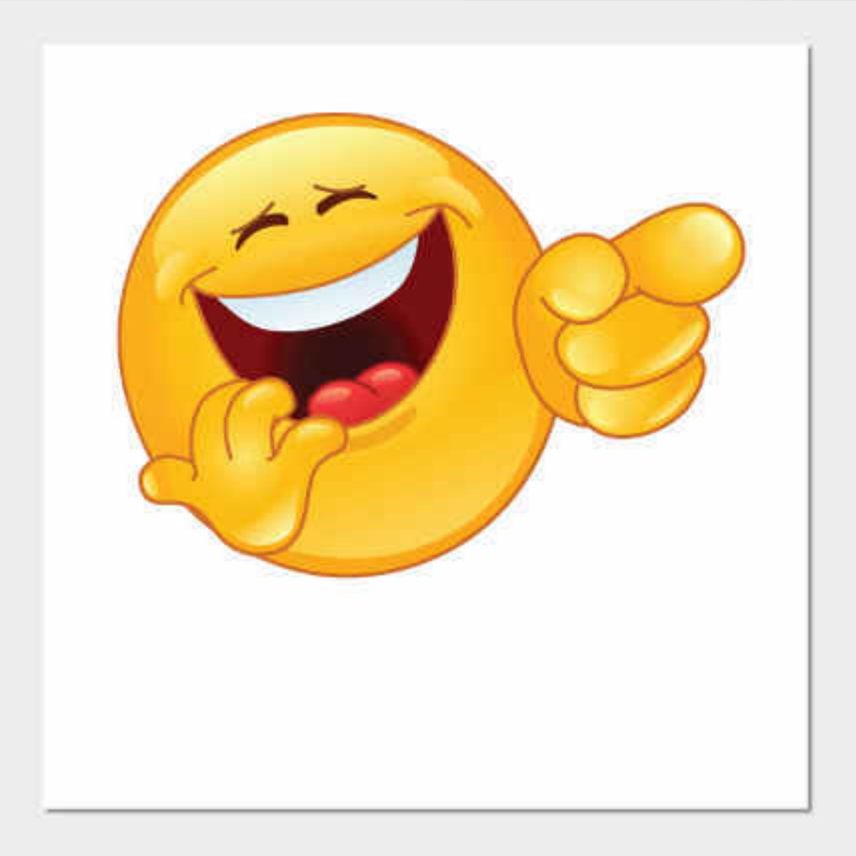
$$\int_0^\infty \frac{sin(s^2)}{s} 2sds \ = \int_0^\infty \sin\,(s^2)2ds \ = \sqrt{\frac{\pi}{2}}$$

Therefore, we are left with the solution

$$A_1 = A_2 = \int_0^\infty \cos(s^2)\,ds = \int_0^\infty \sin(s^2)\,ds = \frac{1}{2}\sqrt{\frac{\pi}{2}}$$

Maths Jokes

The mathematician whose best friend is an abacus has a friend he can count on.



To a mathematician, real life is a special case.

Maths is like marriage – a simple idea, but it can get complicated.

Do you know that 69.846743% of all statistics reflect an unjustified level of precision?

Did you hear about the mathematician who loved his wife so much that he almost told her?

Maths Riddles

1) What did 2 say to the other prime numbers?

2) How can you tell if a mathematician is an extrovert?

3) What is the difference between an argument and a proof?

4) How are mathematicians like the Air Force?

5) What is yellow and differentiable?

6) How many maths professors does it take to change a light bulb?

7) How many number theorists does it take to change the light bulb?

Pell's Equation

Let D be a positive, square-free integer. The Pell's equation (named after John Pell because of an error in attribution by Euler) is an expression of the form

$$X^2 - DY^2 = 1$$

where X and Y are integers.

Consider the equation $x^2 + y^2 = z^2$.We know, for example, (3, 4, 5) is a solution to this equation. Let

$x = (m^2 - n^2), \qquad y = (2mn), \qquad z = (m^2 + n^2)$

$m > n > 0, \quad \gcd(m,n) = 1, \quad m \,\&\, n$ are opposite in parity.

Then

$x^2 + y^2 = z^2$ becomes $(m^2 - n^2)^2 + (2mn)^2 = (m^2 + n^2)^2$

Without loss of generality equation

$x^2 + ay^2 = z^2$ has solutions : $\left(d(m^2 - an^2), 2dmn, d(m^2 + an^2)\right)$

Consider the identity

$$(x^2 - Dy^2)(p^2 - Dq^2) = (xp + Dyq)^2 - D(xq + yp)^2 \qquad (*)$$

If $(p^2 - Dq^2) = 1$

and if (x, y, z) is a solution and if $x^2 - Dy^2 = z^2$ then

$(xp + Dyq,\ xq + yp,\ z)$ is also a solution. To see this, notice if $(p^2 - Dq^2) = 1$ then (*) becomes

$$(x^2 - Dy^2)(1) = (xp + Dyq)^2 - D(xq + yp)^2$$

$$\left(x^2 - Dy^2\right) = \left(xp + Dyq\right)^2 - D\left(xq + yp\right)^2$$

and sub $x^2 - Dy^2 = z^2$

$$z^2 = \left(xp + Dyq\right)^2 - D\left(xq + yp\right)^2$$

$z^2 = X^2 + Y^2$, Hence the solutions are
$X = xp + Dyq, \quad Y = xq + yp, \quad z = z$

An Example:
$x^2 - 5y^2 = 1$
Trivial solution is $(\pm 1, 0)$
one of the non-trivial solution (which is easy to see) (9, 4).
Are there any more solutions to this equation? If so, how many? To answer these questions consider a more general form of this equation.

$$x^2 - Dy^2 = 4\sigma, \qquad \sigma \in \{-1, 1\} \qquad (**)$$

Recall the identity $\left(x^2 - Dy^2\right)\left(p^2 - Dq^2\right) = \left(xp + Dyq\right)^2 - D\left(xq + yp\right)^2$

Sub $x^2 - Dy^2 = 4\sigma$ and $\left(p^2 - Dq^2\right) = 1$ in, we get

$$(4\sigma)(1) = \left(xp + Dyq\right)^2 - D\left(xq + yp\right)^2$$

$$4\sigma = \left(xp + Dyq\right)^2 - D\left(xq + yp\right)^2$$

$$(\pm 2)^2 = \left(xp + Dyq\right)^2 - D\left(xq + yp\right)^2$$

$$\pm 1 = \left(\frac{xp + Dyq}{2}\right)^2 - D\left(\frac{xq + yp}{2}\right)^2$$

which is true for both -1 and 1. Compare this equation with

$z^2 = x^2 + y^2$ and let $x \to x_1, \quad p \to x_2, \quad y \to y_1 \quad and \quad q \to y_2$

and if x_1, x_2, y_1, y_2 are a solution to (**), we see that, given

$x_1 \neq ax_2$ and $y_1 \neq -ay_2$ where $a \in \{-1,1\}$ then

$$x_3 = \frac{x_1x_2 + Dy_1y_2}{2} \quad and \quad y_3 = \frac{x_1y_2 + x_2y_1}{2}$$

is also a solution.

If x_1, x_2, y_1, y_2 are a solution to (**), define

$$\lambda_1 = \left(\frac{x_1 + y_1\sqrt{D}}{2}\right) \text{ and } \lambda_2 = \left(\frac{x_2 + y_2\sqrt{D}}{2}\right) \text{ then}$$

$$\lambda_1 \cdot \lambda_2 = \left(\frac{x_1 + y_1\sqrt{D}}{2}\right)\cdot\left(\frac{x_2 + y_2\sqrt{D}}{2}\right) = \left(\frac{x_1x_2 + Dy_1y_2}{4} + \frac{(x_1y_2 + x_2y_1)\sqrt{D}}{4}\right)$$

$$= \left(\frac{x_3 + y_3\sqrt{D}}{2}\right) = \lambda_3$$

Notice $\lambda's$ are distinct.

If $\left(\frac{x_i + y_i\sqrt{D}}{2}\right)$ is any solution to (**), then $\left(\frac{x_i + y_i\sqrt{D}}{2}\right) = \pm\xi^n$ where ξ is the minimal/fundamental solution.

Non-trivial solutions to Pell's Equation.

Finally, if $x_1, x_2, x_3, y_1, y_2, y_3, \lambda_1, \lambda_2$ defined as above and if (x_1, y_1, σ_1) is any solution to $(**)$ then we can produce infinitely many solutions by

$$(x_n, y_n, \sigma^n),\ (n = 1,2,3,....), \qquad \lambda_1^n = \left(\frac{x_n + y_n\sqrt{D}}{2}\right)$$

Example:

$x^2 - 5y^2 = 1$, we know (9, 4) is a solution. The next solution is:

$$\lambda_1^2 = \left(\frac{x_1 + y_1\sqrt{5}}{2}\right)^2 = \left(\frac{x_2 + y_2\sqrt{5}}{2}\right) \Rightarrow \left(\frac{9+4\sqrt{5}}{2}\right)^2 = \left(\frac{161+72\sqrt{5}}{4}\right)$$

$= \left(\frac{x_2 + y_2\sqrt{5}}{2}\right)$ *hence* the second solution is: (161, 72)

The next solution can be obtained either

$$\lambda_1 \cdot \lambda_2 = \lambda_3 = \lambda_1^3$$

Intuition:

If you haven't noticed yet, an equation of the form

$x^2 - dy^2 = 1$, can be written as $\left(x + y\sqrt{d}\right)\left(x - y\sqrt{d}\right) = 1$ Further notice

$\left(m + n\sqrt{d}\right) + \left(p - q\sqrt{d}\right) = A + B\sqrt{d}$ and

$\left(m + n\sqrt{d}\right) \cdot \left(p - q\sqrt{d}\right) = P + Q\sqrt{d}$

$a + b\sqrt{d}$ is close under $+, \cdot$

That is to say, forms a ring $\mathrm{Z}\left(\sqrt{d}\right)$.

In other words, solving Pell equations in equivalent to finding $z \in \mathrm{Z}\left(\sqrt{d}\right) : \mathrm{N}(z) = 1$, N is the norm. Hence using the cyclic group with a known generator, we can 'jump' from

$z_1 \in \mathrm{Z}\left(\sqrt{d}\right) : \mathrm{N}(z) = 1 \quad to \quad z_2 \in \mathrm{Z}\left(\sqrt{d}\right) : \mathrm{N}(z) = 1.$

and so on.

Using infinite cyclic group we can solve the Pell's Equations.

$$x^2 - 14y^2 = 1$$

$$\sqrt{14} = 3 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{3 + \sqrt{14}}}}}$$

$3 + \sqrt{14}$ is purely periodic with period length 4. Truncating the expansion at the end of the first period, we get

$$3 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{\cfrac{1}{1}}}} = \frac{15}{4} \approx \sqrt{14}$$

The numerator and denominator is the fundamental solution to the equation above. $\lambda_1 = \left(\frac{15 + 4\sqrt{14}}{2}\right), so$ the fundamental solution is $(15, 4)$

Infinite continued fraction.

ANSWERS:

1)

$$\sqrt{11} = [3; 3,6,3,6, \dots] = 3 + \cfrac{1}{3 + \cfrac{1}{6 + \cfrac{1}{3 + \cfrac{1}{6 + \cfrac{1}{3 + \cdots}}}}}$$

2)

$$\varphi = \frac{1+\sqrt{5}}{2} = [1; 1,1,1, \dots] = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}}}}$$

3)

$$e = [2; 1,2,1,1, \dots] = 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{4 + \cdots}}}}}$$

ANSWERS to Maths Riddles

1) ”I’m glad that one of us is even-tempered”.

2) When talking to you, He looks at your shoes.

3) An argument will convince a reasonable man, but proof is needed to convince an unreasonable one.

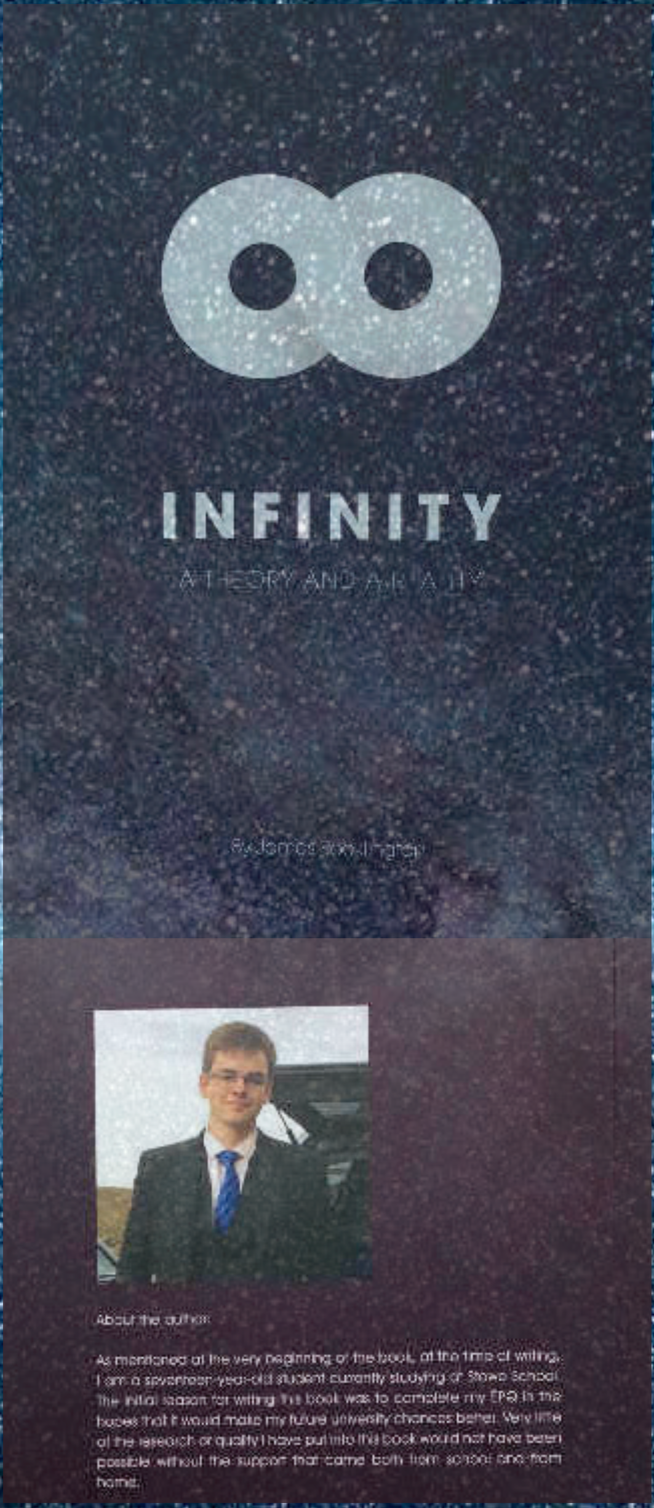
4) They both use (pi) lots. (pilots!)

5) Bananalytic function.

6) Just one. But he needs the help of six research students, three programmers, two post-docs, and a secretary.

7) No one knows the exact number, but it is believed to be an elegant prime.

We would like to congratulate James Pocklington on publishing his first book 'Infinity'.



This book is about maths, physics, religion, biology, and life. In other words, this book is about everything, 'since everything is a small part of infinity'!

As well as his inspiring views, James has reached others including, Nigel Farage, Sir Richard Branson, and asked their opinions on infinity.

If you would like to have a hard copy of this book, please email James on pocklingtonjh@gmail.com