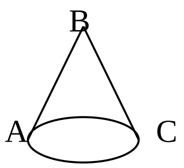
Section 1: Matrix arithmetic

Notes and Examples

These notes contain subsections on
- **Introducing matrices**
- **Multiplying a matrix by a scalar**
- **Multiplying matrices**
- **The identity matrix**

Introducing matrices

A matrix is simply a way of storing information. For example, the diagram below shows a map of the roads linking three towns A, B and C. The corresponding '*direct route*' matrix is shown beside it.



$$\begin{array}{c} \\ A \\ B \\ C \end{array} \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix} \end{array}$$

In this section you learn to multiply a matrix by a number and to multiply two matrices.

Matrices are classified by number of rows and the number of columns they have. The matrix above has 3 rows and 3 columns, it is a 3×3 matrix (read as '3 by 3').

A matrix with m rows and n columns is an $m \times n$ matrix. This is called the **order** of the matrix.

A **square matrix** is a matrix with the same number of rows as columns.

Multiplying a matrix by a scalar

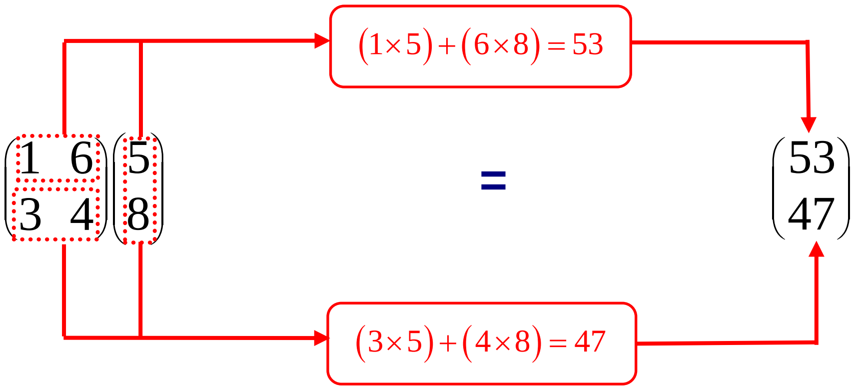
A matrix can be multiplied by a scalar (a number). Each element of the matrix is multiplied by the scalar.

For example, $\mathbf{A} = \begin{pmatrix} 2 & -1 \\ 3 & 0 \end{pmatrix} \Rightarrow 3\mathbf{A} = \begin{pmatrix} 6 & -3 \\ 9 & 0 \end{pmatrix}$

Multiplying matrices

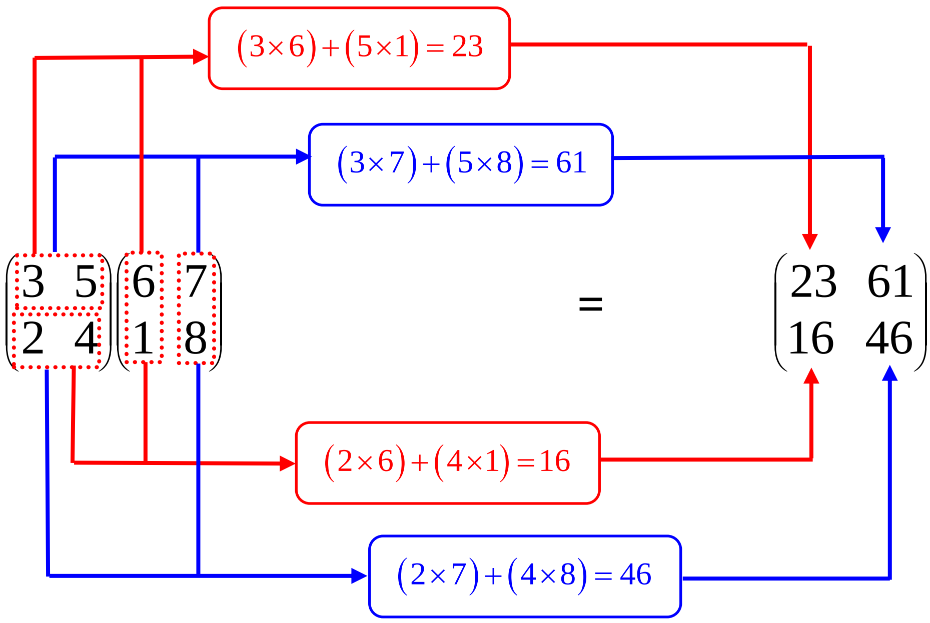
Multiplying matrices is an important skill which you must master. It takes a bit of getting used to, but after plenty of practice you will find it quite straightforward.

The diagram below shows the process of multiplying a 2 × 2 matrix by a 2 × 1 matrix.



So $\begin{pmatrix} 1 & 6 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 5 \\ 8 \end{pmatrix} = \begin{pmatrix} 53 \\ 47 \end{pmatrix}$

The diagram below shows the steps used when multiplying a 2 × 2 matrix by another 2 × 2 matrix.



So $\begin{pmatrix} 3 & 5 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} 6 & 7 \\ 1 & 8 \end{pmatrix} = \begin{pmatrix} 23 & 61 \\ 16 & 46 \end{pmatrix}$

A similar technique applies to all matrix multiplications. You use each row of the first (i.e. left) matrix with each column, in turn, of the second matrix.

The important points to remember are:

- Use each row of the first matrix with each column of the second.
- When you are using row a of the first matrix with column b of the second matrix, the result gives you the element in row a, column b of the product matrix.
- To multiply matrices, the number of columns in the first matrix must be the same as the number of rows in the second matrix. If this is not the case, the matrices do not conform and cannot be multiplied.

You only need to be able to multiply a 2×2 matrix by another 2×2 matrix or by a 2×1 matrix.

Example 1

A is the matrix $\begin{pmatrix} 2 & 3 \\ -1 & 5 \end{pmatrix}$.

B is the matrix $\begin{pmatrix} -3 & 0 \\ 1 & 4 \end{pmatrix}$.

C is the matrix $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$.

Find:

(i) **AB** (ii) **BA**

(iii) **AC** (iv) **BC**

Solution

(i) $\mathbf{AB} = \begin{pmatrix} 2 & 3 \\ -1 & 5 \end{pmatrix}\begin{pmatrix} -3 & 0 \\ 1 & 4 \end{pmatrix} = \begin{pmatrix} 2\times -3 + 3\times 1 & 2\times 0 + 3\times 4 \\ -1\times -3 + 5\times 1 & -1\times 0 + 5\times 4 \end{pmatrix}$

$$= \begin{pmatrix} -3 & 12 \\ 8 & 20 \end{pmatrix}$$

(ii) $\mathbf{BA} = \begin{pmatrix} -3 & 0 \\ 1 & 4 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ -1 & 5 \end{pmatrix} = \begin{pmatrix} -3\times 2 + 0\times -1 & -3\times 3 + 0\times 5 \\ 1\times 2 + 4\times -1 & 1\times 3 + 4\times 5 \end{pmatrix}$

$$= \begin{pmatrix} -6 & -9 \\ -2 & 23 \end{pmatrix}$$

(iii) $\mathbf{AC} = \begin{pmatrix} 2 & 3 \\ -1 & 5 \end{pmatrix}\begin{pmatrix} 3 \\ -2 \end{pmatrix} = \begin{pmatrix} 2\times 3 + 3\times -2 \\ -1\times 3 + 5\times -2 \end{pmatrix}$

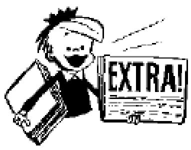
$$= \begin{pmatrix} 0 \\ -13 \end{pmatrix}$$

(iv) $\mathbf{BC} = \begin{pmatrix} -3 & 0 \\ 1 & 4 \end{pmatrix}\begin{pmatrix} 3 \\ -2 \end{pmatrix} = \begin{pmatrix} -3\times3+0\times-2 \\ 1\times3+4\times-2 \end{pmatrix}$

$$= \begin{pmatrix} -9 \\ -5 \end{pmatrix}$$

Note that it is not necessary to write out the calculations in full, as in the example above. It is shown here so that you can see what is being done. You may like to write it out in full until you feel confident, or you may feel able to miss out that step from the beginning.

Notice the important point that matrix multiplication, unlike the multiplication of numbers, is not commutative: i.e. **AB** $\neq$ **BA**.



In addition, matrix multiplication has the following properties:

- Matrix multiplication is associative
- Matrix multiplication is distributive

The identity matrix

The matrix **I** = $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is called the 2×2 **identity matrix** because when you multiply any 2×2 matrix **A** by **I** you get **A** as the answer.
I acts like the number 1 in the multiplication of numbers.

This means that for any 2×2 matrix **A**: **IA** = **AI** = **A.**