Section 2: Matrix transformations

Notes and Examples

These notes contain subsections on

Transformations

Matrices have many applications. This chapter looks at their use to describe transformations.

A linear transformation is a transformation in which the image (x', y') of a point (x, y) can be written as

$$x' = ax + by$$
$$y' = cx + dy$$

This can be written in the form of a matrix equation:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

The matrix form makes it easy to find the image of a point using matrix multiplication.

Example 1

Find the images of the points A (3, 1), B (-2, 4) and C (5, -1) under the transformation represented by the matrix $\begin{pmatrix} 2 & 3 \\ -1 & 0 \end{pmatrix}$.

Solution

For A: $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 9 \\ -3 \end{pmatrix}$

The image of A is (9, -3).

For B: $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} -2 \\ 4 \end{pmatrix} = \begin{pmatrix} 8 \\ 2 \end{pmatrix}$

The image of B is (8, 2)

For C: $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 5 \\ -1 \end{pmatrix} = \begin{pmatrix} 7 \\ -5 \end{pmatrix}$

The image of C is (7, -5)

AQA FM Matrices 2 Notes and Examples

By the time you have worked through this section, you should be familiar with the matrices for simple transformations such as reflections in the x axis, the y axis, the line $y = x$ and the line $y = -x$, rotations about the origin through 90° and 180, and enlargements.

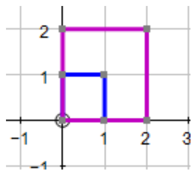
An easy way to find the matrices representing simple transformations is to think about the images of the points (1, 0) and (0, 1).

Under the transformation $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$:

- The image of the point (1, 0) is $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix}$, which is the first column of the matrix.
- The image of the point (0, 1) is $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} b \\ d \end{pmatrix}$, which is the second column of the matrix.

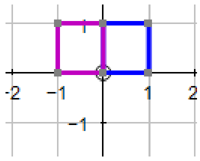
Enlargements

An enlargement, centre the origin, with scale factor k maps the point (1, 0) to the point (k, 0) and the point (0, 1) to the point (0, k).
So, the matrix representing this transformation is $\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$.



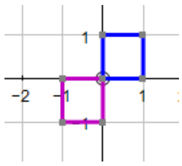
AQA FM Matrices 2 Notes and Examples

Rotations about the origin



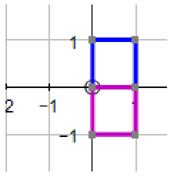
A rotation through 90° anticlockwise about the origin maps the point (1, 0) to the point (0, 1) and the point (0, 1) to the point (-1, 0).

So the matrix representing this transformation is $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.



A rotation through 180° about the origin maps the point (1, 0) to the point (-1, 0) and the point (0, 1) to the point (0, -1).

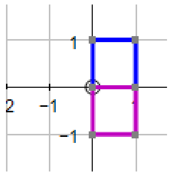
So the matrix representing this transformation is $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$.



A rotation through 90° clockwise about the origin maps the point (1, 0) to the point (0, -1) and the point (0, 1) to the point (1, 0).

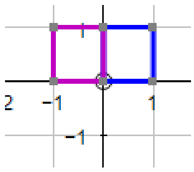
So the matrix representing this transformation is $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Reflections



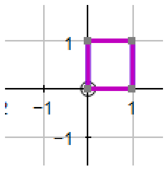
Reflection in the x-axis leaves the point (1, 0) unchanged but maps the point (0, 1) to the point (0, -1).

So the matrix representing this transformation is $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.



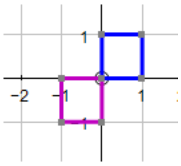
Reflection in the y-axis maps the point (1, 0) to the point (-1, 0) but leaves the point (0, 1) unchanged.

So the matrix representing this transformation is $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$.



Reflection in the line $y = x$ maps the point (1, 0) to the point (0, 1) and maps the point (0, 1) to the point (1, 0).

So the matrix representing this transformation is $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.



Reflection in the line $y = -x$ maps the point (1, 0) to the point (0, -1) and maps the point (0, 1) to the point (-1, 0).

So the matrix representing this transformation is $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$.

For some practice in recognising matrix transformations, try the activity ***Matrix matchings***.

Composite transformations

Suppose you want to find the image of a point (x, y) under transformation P followed by transformation Q.

You will start by finding the image under P by working out $\mathbf{P}\begin{pmatrix} x \\ y \end{pmatrix}$.

Then you need to apply transformation Q, by multiplying Q by the result of the first transformation. So this will give $\mathbf{Q}\left(\mathbf{P}\begin{pmatrix} x \\ y \end{pmatrix}\right)$. This can also be written as

$\mathbf{QP}\begin{pmatrix} x \\ y \end{pmatrix}$. So the result of carrying out transformation P followed by transformation Q is a transformation represented by the matrix **QP**.

Example 2

(i) Write down the matrix **A** which represents a rotation of 90° clockwise about the origin.

(ii) Write down the matrix **B** which represents a reflection in the line $y = x$.

(iii) Find the matrix which represents the transformation **A** followed by **B**. Describe this transformation.

Solution

(i) $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

(ii) $\mathbf{B} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

(iii) $$\mathbf{BA} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

This is a reflection in the x-axis.